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A SURVEY OF STATISTICAL METHODS IN SYSTEMS RELIABILITY USING BERNOULLI SAMPLING OF COMPONENTS

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ABSTRACT

A survey of the literature on statistical inference concerning systems reliability using Bernoulli sampling of components is provided.

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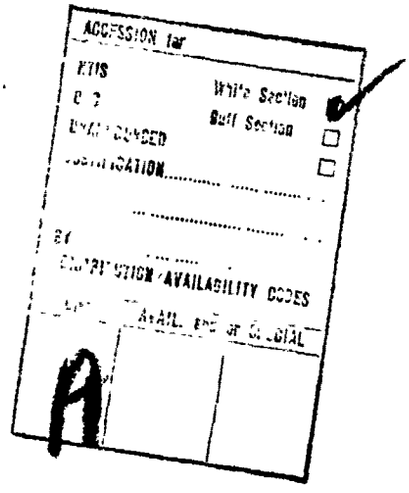

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A SURVEY OF STATISTICAL METHODS IN SYSTEMS RELIABILITY USING BERNOULLI SAMPLING OF COMPONENTS

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1. Introduction. A fundamental problem in the area of statistical inference in reliability theory is the statistical estimation of the reliability of a system from experimental data collected on each of the individual components. In this survey, I will try to describe the state of the art as I envision it. As I will subsequently demonstrate, much still remains to be accomplished in this area, particularly in the construction of statistical methods which are satisfactory when the true system reliability is high.

The paper is divided into sections as follows. In Section 2 we describe coherent systems and list some of their basic properties. In Section 3, we discuss various measures of the importance of individual components in a system. The fourth section treats methods of statistical inference for series systems, and parallel systems are discussed in Section 5. The sixth section is devoted to arbitrary coherent systems and the seventh section is concerned with Bayesian methods. A discussion of the use of loss functions is treated in Section 8. Some concluding remarks are made in the ninth section.

2. Coherent systems. We now introduce some definitions and notation which will be used throughout this survey.

A system is regarded as a specified configuration of components. To each component c, we associate the variable x_c, where x_c assumes the value 0

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or 1 . A system of k components is then defined as a function $\varphi: \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_k \to \{0,1\}$, where each $\mathcal{X}_i = \{0,1\}$ and $x_i = x_{c_i} \in \mathcal{X}_i$, $i = 1, 2, \ldots, k$.

In the customary language which originates from the engineering applications which have motivated the study of reliability theory,

$x_i = 1$ means the component is functioning,

$x_i = 0$ means the component has failed,

and

$$\varphi(x_1, x_2, \ldots, x_k) = \begin{cases} 1 & \text{means the system is functioning} \\ 0 & \text{means the system has failed.} \end{cases}$$

The system φ is said to be a coherent system if

1. $\varphi(0, 0, \ldots, 0) = 0$,
2. $\varphi(1, 1, \ldots, 1) = 1$, and
3. $\varphi(x_1, x_2, \ldots, x_k)$ is increasing in each argument.

We let $\tilde{x} = (x_1, x_2, \ldots, x_k)$. Then we can define the following types of coherent systems.

1. Series systems, $\varphi(\tilde{x}) = \prod_{i=1}^{k} x_i$,
2. Parallel systems, $\varphi(\tilde{x}) = 1 - \prod_{i=1}^{k} (1 - x_i)$,
3. Series-parallel systems, $\varphi(\tilde{x}) = \prod_{j=0}^{r-1} \left(1 - \prod_{i=k_j+1}^{k_{j+1}} (1 - x_i) \right)$, where $k_0 = 0$ and $\sum_{j=0}^{r-1} k_j = k$.

4. Parallel-series system, $\varphi(\tilde{x}) = 1 - \prod_{j=0}^{r-1}\left(1 - \prod_{i=k_j+1}^{k_{j+1}} x_i\right)$, where $k_0 = 0$ and $\sum_{j=0}^{r-1} k_j = k$.

5. r out of k systems, $\varphi(\tilde{x}) = \sum_j \prod_{i\in A_j} x_i \prod_{i\in A_j^c}(1 - x_i)$, where A_j is any subset of $\{1, 2, \ldots, k\}$ with $|A_j| \geq r$, where $|A|$ denotes the cardinality of the set A, and the sum is over all such subsets.

A series-parallel system consists of k components divided into disjoint subsets, each subset is a parallel system and the parallel systems, regarded as individual components, are combined into a series system.

The parallel-series system is analogously defined with each subset being a series system and the totality is combined into a parallel system.

An r out of k system functions if and only if at least r of the k components function.

There is an extensive literature on the theory of coherent systems. These ideas were first formulated in Z. W. Birnbaum, J. D. Esary, and S. C. Saunders [7], in which they extended some ideas of E. F. Moore and C. E. Shannon [29]. For many further details, the reader is referred to R. E. Barlow and F. Proschan [2], [4] and James P. Lipp [18].

Now let X_i, $i = 1, 2, \ldots, k$ be a family of mutually independent Bernoulli random variables, and $P\{X_i = 1\} = p_i = 1 - P\{X_i = 0\}$, $i = 1, 2, \ldots, k$. Then we define the reliability function $h(p_1, p_2, \ldots, p_k)$ of the system φ as

$$h(\tilde{p}) = E\{\varphi(X_1, X_2, \ldots, X_k)\} = P\{\varphi(X_1, X_2, \ldots, X_k) = 1\} \tag{1}$$

where $\tilde{p} = (p_1, p_2, \ldots, p_k)$. By (1), we see that $h(\tilde{p})$ is the probability that the system is functioning, which coincides with the interpretation of the term reliability in the customary engineering context.

It is easily seen that for the series, parallel, series-parallel, parallel-series and r out of k systems,

(2) $$h(\tilde{p}) = \varphi(\tilde{p}) \ ,$$

upon extending the domain of φ to the k-fold product of unit intervals and extending the range of φ to the unit interval. On the set $\{0 < p_i < 1,\ i = 1, 2, \ldots, k\}$, $h(\tilde{p})$ is increasing in each p_i, $i = 1, 2, \ldots, k$, whenever φ is a coherent system.

The techniques to be described in this survey are all concerned with methods for statistical inference concerning $\varphi = h(\tilde{p})$.

Before turning to the problems of statistical inference, I would like to call attention to some concepts which I feel have potential utility in statistical inference for systems reliability, but thus far appear not to have been really exploited.

3. The relative importance of individual components in a system. Z. W. Birnbaum [6] defined two notions of the importance of a component in a system. The reliability importance of component i, $1 \leq i \leq k$ is defined as

(3) $$I_{h,i}(\tilde{p}) = \frac{\partial h(\tilde{p})}{\partial p_i}$$

and the structural importance of component i is defined as

(4) $$I_h(i) = \left.\frac{\partial h(\tilde{p})}{\partial p_i}\right|_{p_1 = \ldots = p_k = \frac{1}{2}} .$$

In R. E. Barlow and F. Proschan [3], the notion of a fault tree was employed to justify a different definition of the importance of a particular component in a system. In the notation and terminology of this paper, this notion of the importance of the ith component can be interpreted as the

probability that component i caused the failure of the system given that the system has failed. In R. E. Barlow and F. Proschan [3], this concept was introduced for continuous life distributions, rather than the model employed here in terms of Bernoulli random variables. Thus for Barlow and Proschan the probability that more than one component caused the failure of the system is zero. However, for the Bernoulli model of this paper, such is not the case. Therefore, some modifications of this definition are needed. Hence, we would like to propose the following ad hoc alternative in this case and define the importance of component i by

$$(5)\quad I_i = \frac{1}{P\{\varphi(\tilde{X})=0\}} \sum_{j=1}^{k} \frac{1}{j} \sum_{1\le \ell_1<\ldots<\ell_{j-1}\le k,\ \ell_m \neq i,\ 1\le m\le j-1,} P\{\varphi(\tilde{X})=0,\ X_i = X_{\ell_1} = X_{\ell_2} = \ldots = X_{\ell_{j-1}} = 0\}.$$

This is equivalent to assigning the cause of failure to a single component at "random" when more than one component has failed and the system has failed. The result obtained using (5) will not in general coincide with the corresponding result that would be obtained if specific lifetime distributions for the individual components were assumed.

For example, if the system is a series system with two components, then the component which causes the failure is the first component to fail. If Y and Z are the waiting times to failure and $F(y)$ and $G(z)$ are the respective waiting time distributions, then the probability that the first component causes the system failure, given that both components have failed on or before time T is given by

$$P\{Y < Z | Y \leq T, Z \leq T\} = [F(T)G(T)]^{-1} \int_0^T F(z)dG(z) .$$

It is easily seen that in general $P\{Y > Z | Y \leq T, Z \leq T\} \neq \frac{1}{2}$; however, if $F(y) = G(y)$, $y > 0$ or if $F(y)$ and $G(z)$ are uniform distributions on $[0, u_y]$, $[0, u_z]$ respectively with $T \leq \min[u_y, u_z]$, then $P\{Y < Z | Y \leq T, Z \leq T\} = \frac{1}{2}$.

Still another alternative for a notion of the importance of the ith component is suggested by the following considerations. In many applications, the objective is to produce a system of hign reliability, hence it is natural to suppose that the reliability of each component should be high. In this case, improvement in the system reliability is affected principally by the behavior of $h(p_1, p_2, \ldots, p_k)$ as each p_i tends to unity. Thus, a further suggestion is to define the importance of the ith component by

$$I_i^* = \left. \frac{\partial h(\tilde{p})}{\partial p_i} \right|_{p_1 = \ldots = p_k = 1} , \tag{6}$$

where the indicated differentiation is to be interpreted as a left-derivative.

It is hoped that these suggestions prove to be useful. As of this date, I have not yet explored their properties or their applications to statistical methods of assessing reliability.

4. Methods for statistical inference for the reliability of series systems. To simplify notation, we define $q_i = 1 - p_i$, $i = 1, 2, \ldots, k$. Then q_i is the probability that the ith component is defective. Assume that N_i Bernoulli trials have been made on the ith component. Then, if X_i is the number of successes observed on the ith component, we have

$$(7) \qquad P\{X_1 = x_1, \ldots, X_k = x_k\} = \prod_{i=1}^{k} \binom{N_i}{x_i} p_i^{x_i} q_i^{N_i - x_i}, \quad x_i = 1, 2, \ldots, N_i ,$$

$i = 1, 2, \ldots k$.

Clearly $(X_1, X_2, \ldots, X_k)$ is a sufficient statistic and any statistical technique need employ only the data set $\{(x_1, N_1), (x_2, N_2), \ldots, (x_k, N_k)\}$.

We first treat the case of a series system. Then since

$$h(\tilde{p}) = \prod_{i=1}^{k} p_i ,$$

a natural choice as an estimator of $h(\tilde{p})$ is

$$(8) \qquad h(\hat{p}) = \prod_{i=1}^{k} (x_i / N_i) = \prod_{i=1}^{k} (\hat{p}_i) ,$$

which is both the maximum likelihood estimator and the minimum variance unbiased estimator. However, this estimator can have some undesirable properties. In the case of parallel systems, its behavior as a point estimator can be very poor and we present an example to this effect in Section 5. Many writers have attempted to exploit (8) as a basis for determining a lower confidence limit $\underline{L}_h$ to $h(\tilde{p})$. Naturally, all confidence limits to be described depend on the confidence coefficient $1 - \alpha$; the dependence on α will be suppressed in the notation that will be employed.

We now present a summary of various techniques that have been proposed.

F. Nishime [32] computes the lower confidence limit $\underline{L}_i$ for each p_i using the binomial distribution. Let

$$\underline{L}^* = \frac{1}{k} \sum_{i=1}^{k} \underline{L}_i$$

and let $\underline{L}'$ be the lower confidence limit for the binomial parameter computed from the binomial distribution with $N = \sum_{i=1}^{k} N_i$ and $x = \sum_{i=1}^{k} x_i$. Then

$$\underline{L}_h = (\underline{L}'/\underline{L}^*)^k \prod_{i=1}^{k} \underline{L}_i \quad .$$

The rationale for the method is by no means obvious. If the N_i are very disparate, this can give unreasonable answers.

N. R. Garner and R. W. Vail, Jr. [13] have proposed another method of combining component data to obtain a lower confidence limit. Let $\underline{N} = \min_{1 \le i \le k} N_i$ let $y^* = \sum_{i=1}^{k} (N_i - x_i)$, that is, y^* is the total number of failures observed. Further let $\bar{y} = \max_{1 \le i \le k} (N_i - x_i)$. Then if $y^* \le \underline{N}$, the lower confidence limit is given by

$$(9) \qquad \underline{L}_h = \sum_{z=\underline{N}-y^*}^{\underline{N}-\bar{y}} L^{(\underline{N})}(z)\, b(z;\underline{N}, h(\hat{p})) \Big/ \sum_{w=\underline{N}-y^*}^{\underline{N}-\bar{y}} b(w;\underline{N}, h(\hat{p})) \quad ,$$

where $b(x;M,p)$ is the binomial probability function with parameters M and p and $L^{(N)}(x)$ is the lower confidence limit for the binomial parameter when x successes are observed in $\underline{N}$ Bernoulli trials. It can easily be seen that this produces optimistic (non-conservative) answers in some cases.

D. L. Lindstrom and J. H. Madden, (see D. K. Lloyd and M. Lipow [19]) have proposed the following procedure. Interpret $\underline{N}\, h(\hat{p})$ as the number of successes in $\underline{N}$ Bernoulli trials. That is, $\underline{N}\, h(\hat{p})$ is interpreted as the

observation from a binomial distribution with parameters $(\underline{N}, h(\tilde{p}))$ and this is employed to obtain a lower confidence limit $\underline{L}_h$ for $h(\tilde{p})$.

R. J. Buehler [10] suggested a technique for obtaining "optimal" one-sided confidence intervals for the product of binomial parameters. His technique, however, requires that the possible observed k-tuples be ordered. Thus, if $k = 2$, there are $(N_1+1)(N_2+1)$ sample points that must be ordered in a sequence. Specifically, let $c_{N_1, N_2}(x_1, x_2)$ satisfy

$$P\{c_{N_1, N_2}(x_1, x_2) \leq p_1 p_2 \leq 1\} \geq 1 - \alpha \quad . \tag{10}$$

Then the numbers $c_{N_1, N_2}(x_1, x_2)$ form a set of lower confidence limits for $p_1 p_2$. These are to be ordered so that

$$c_{N_1, N_2}(x_1^{(j)}, x_2^{(j)}) \leq c_{N_1, N_2}(x_1^{(i)}, x_2^{(i)}) \tag{11}$$

whenever $i < j$, where $1 \leq i < j \leq (N_1+1)(N_2+1)$. The unique sequence such that the $c_{N_1, N_2}(x_1, x_2)$ are uniformly largest subject to the specific ordering chosen is given by

$$c_{N_1, N_2}(x_1^{(i)}, x_2^{(i)}) = \inf\{p_1 p_2 : \sum_{j \geq i} b(N_1, p_1 : x_1^{(j)})\, b(N_2, p_2 : x_2^{(j)}) > \alpha\} \quad , \tag{12}$$

where the index j is determined by the ordering. Since (12) depends on the choice of ordering, the question of how a particular ordering should be chosen is by no means clear. Buehler has suggested the ordering based on the criterion

$$c_{N_1}(x_1, \sqrt{1-\alpha})\, c_{N_2}(x_2, \sqrt{1-\alpha}) \leq c_{N_1}(x_1', \sqrt{1-\alpha})\, c_{N_2}(x_2', \sqrt{1-\alpha})$$

implies

$$c_{N_1, N_2}(x_1, x_2) \leq c_{N_1, N_2}(x_1', x_2') \ ,$$

where $c_{N_1}(x, \sqrt{1-\alpha})$ denotes the lower confidence limit for a single binomial parameter with confidence coefficient $\sqrt{1-\alpha}$.

Under suitable conditions the binomial distribution may be replaced by the Poisson distribution, which offers some computational advantages.

In general, Buehler's method is difficult to utilize, particularly for $k > 2$, because of the computational effort involved in ordering the sample outcomes.

J. L. Epstein [12] considered the case $k = 2$ and compared the estimators $\hat{h}_1 = (X_1 + 1)(X_2 + 1)/N_1N_2$ and $\hat{h}_2 = X_1X_2/N_1N_2$. Let $T_i = \{(x_1, x_2): N_1N_2\hat{h}_1 = i\}$ and $S_j = \{(x_1, x_2): N_1N_2\hat{h}_2 = j\}$, $1 \leq i \leq (N_1+1)(N_2+1)$, $0 \leq j \leq N_1N_2$. Then the sets T_i and S_j determine partitions of the sample space which we denote by $\mathfrak{T}$ and $\mathfrak{S}$ respectively. The sets T_i and S_j are ordered so that a monotonicity condition is satisfied, that is if $(x_1, x_2) \in \bigcup_{i=1}^{k} S_i$ then $(y_1, y_2) \in \bigcup_{i=1}^{k} S_i$ for all $y_1 \leq x_1$, $y_2 \leq x_2$. Epstein computes some numerical examples of confidence limits and it appears from his investigation that the partition $\mathfrak{T}$ possesses some advantages over the partition $\mathfrak{S}$.

In an unsigned and undated mimeographed report which has come to my attention (listed in references as Anonymous [1]), the following technique

has been suggested. Let $\hat{p}_i = X_i/N_i$, $i = 1, 2, \ldots, k$ and consider

$$\log \prod_{i=1}^{k} \hat{p}_i = \log h(\hat{p}) = \sum_{i=1}^{k} \log \hat{p}_i \quad .$$

Then since $\sum_{i=1}^{k} \log \hat{p}_i$ is a sum of independent random variables, we assume that the central limit theorem can be employed and use a normal approximation to obtain the lower confidence limit $\underline{L}_h$. Note that the mean and variance of $\log \hat{p}_i$ do not exist. The author suggests using the "relationship between the moments of the normal and the moments of the lognormal distributions to determine the moments of $\log \hat{p}_i$".

One of the techniques in the "folklore" of reliability theory has been the use of Poisson approximations when the reliability of each component can be assumed to be "high". Intuitively, this technique can be sketched as follows. Write

$$1 - h(\tilde{p}) = 1 - \prod_{i=1}^{k} (1 - q_i), \qquad q_i = 1 - p_i, \quad i = 1, 2, \ldots, k \quad .$$

Then set

$$q_i = \lambda_i/N_i, \quad i = 1, 2, \ldots, k \quad .$$

Accordingly, we can write

$$1 - h(\tilde{p}) = 1 - \prod_{i=1}^{k} (1 - \lambda_i/N_i) \sim \sum_{i=1}^{k} \lambda_i/N_i \quad .$$

Thus, if we let $N_i - X_i = Y_i$, $i = 1, 2, \ldots, k$, we can regard the Y_i's as being independently Poisson distributed random variables with parameters λ_i. This idea has been extensively exploited in the statistical literature and we provide three illustrations.

In the book by B. V. Gnedenko, Yu. K. Belyayev and A. D. Solov'yev [14], the following method is suggested, which the authors attribute to R. A. Mirniy [28].

Let $\varphi = -\sum_{i=1}^{k} \log p_i$ and let $\bar{\varphi} = \max(-\sum_{i=1}^{k} \log p_i)$ subject to the conditions $\lambda_i \geq 0$, $i = 1, 2, \ldots, k$ and $\sum_{i=1}^{k} \lambda_i \leq \Delta_{1-\alpha}(Y_1, Y_2, \ldots, Y_k)$, where $P\{\sum_{i=1}^{k} \lambda_i \leq \Delta_{1-\alpha}\} \geq 1-\alpha$; $\Delta_{1-\alpha}$ is the solution in Δ of $P_{\Delta}\{Z \leq \sum_{i=1}^{k} Y_i\} = \alpha$ and Z has the Poisson distribution with parameter Δ. Then they obtain

$$\underline{L}_h = e^{-\bar{\varphi}} , \tag{13}$$

where

$$\bar{\varphi} = \Delta_{1-\alpha}(Y_1, Y_2, \ldots, Y_k)/\underline{N} . \tag{14}$$

I. V. Pavlov [34] provides a comprehensive discussion of Poisson methods for series reliability. L. N. Bol'shev and E. A. Loginov [8] use the Poisson approximation and determine the lower confidence limit by

$$\underline{L}_h = 1 - \frac{1}{2\underline{N}} \chi^2_{2y^*+2}(1-\alpha) , \tag{15}$$

where $\chi^2_{2y^*+2}$ denotes the $(1-\alpha)$ point of the chi-square distribution with $2y^*+2$ degrees of freedom.

The evidence accumulated thus far suggests that these are likely to produce conservative results.

A conservative method in keeping with the above-mentioned "folk theorem" of reliability theory is described in the paper by A. Winterbottom [41].

A method known as the key test results method was introduced by K. A. Weaver [39] and described by him for the case $N_1 = N_2 = \cdots = N_k = N$ and extended to arbitrary N_i, $1 \leq i \leq k$ by A. Winterbottom [40], [41]. Specifically, define

$$G(h(\hat{p}); \tilde{p}) = P_{\tilde{p}} \{h(\prod_{i=1}^{k} (x_i/N_i)) \geq h(\hat{p})\} \quad , \tag{16}$$

that is, the probability of observing an estimator of the form (8) greater than or equal to the observed value $h(\hat{p})$. Then

$$\{\tilde{p}: G(h(\hat{p}); \tilde{p})\} \geq \alpha \tag{17}$$

constitutes the set of parameter points that would be accepted in a significance test of size α. The lower confidence limit is given by $\inf\{h(\tilde{p}): G(h(\hat{p}), \tilde{p}) \geq \alpha\}$.

Winterbottom gives a graphical method for solving (17) when $k = 2$ and comments on the difficulties encountered when $k > 2$.

With no loss of generality, we can let $\underline{N} = N_1 \leq N_2 \leq \ldots \leq N_k$. Then the key test results are those with $0 \leq x_1 \leq N_1$, $x_i = N_i$, $i = 2, \ldots, k$. Using the key test results, Winterbottom shows that

$$\underline{L}_h = \inf\{p: \sum_{z=x_1}^{N_1} b(z; N_1, p) \geq \alpha\} \quad , \tag{18}$$

where, if $h(\hat{p}) = u$, interpret the experimental outcome as $x_1 = \underline{N}u$ and compute the lower confidence limit from (18).

We conclude this section by briefly mentioning the work of C. Mack. In a sequence of four papers, he treated various problems concerned with

determining confidence limits for products of binomial parameters. In the first of these [20], he provides a number of technical improvements to the likelihood ratio technique proposed by Madansky [24]. The third [22] treats the case of the product of two binomial parameters ($k = 2$). The technique is basically a computational method, which can be used when the number of failures is small. Tables of formulas and numerical values are given to facilitate the computation.

5. Methods for statistical inference for parallel systems. For parallel systems, we write $1 - h(\tilde{p}) = \prod q_i$. Thus, there is a natural duality between parallel and series systems and many statistical techniques for series systems may be transformed into equivalent methods for parallel systems upon replacing X_i by $N_i - X_i$, p_i by q_i and conversely. Then lower confidence limits for $\prod_{i=1}^{k} p_i$ become upper confidence limits for $\prod_{i=1}^{k} q_i$. This duality has been exploited in the papers by R. J. Buehler [10], J. L. Epstein [12], Anonymous [1] and C. Mack [20].

Note that in the case of the suggestion by Anonymous [1], if no failures have been observed for any component, then the data cannot be used. This is somewhat contradictory in that one should feel happy that no failures are observed, yet this prevents utilization of the experimental data.

We now consider methods specifically designed for parallel systems.

B. Harris [15] assumed that $Y_1, Y_2, \ldots, Y_k$ are independently Poisson distributed with parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$, where $Y_1, Y_2, \ldots, Y_k$ are to be regarded as the number of observed failures for each of the components. Further λ_i is to be regarded as $N_i q_i$, $i = 1, 2, \ldots, k$. Then, if

(19) $$\theta = \prod_{i=1}^{k} \lambda_i = \prod_{i=1}^{k} (q_i/N_i) \ ,$$

the conditional distribution of Y_1 given $Y_2-Y_1, \ldots, Y_k-Y_1$ depends only on θ and not on the individual λ_i's . This observation may be employed to obtain uniformly most powerful similar tests and uniformly most accurate unbiased confidence intervals for θ . The contitional distribution referred to above is known as the generalized incomplete modified Bessel distribution and is tabulated in B. Harris and A. P. Soms [16].

In the remaining two papers by C. Mack [21], [23], methods for use with parallel systems are described. Both are restricted to the case $k = 2$, the first giving appropriate methods for use when the number of observed failures is small. The second of these is largely devoted to tables to facilitate computation of the upper confidence limit for the probability of failure of the system.

We digress briefly to comment on the point estimation of $h(\tilde{p})$. The minimum variance unbiased estimator and maximum likelihood estimator of $1 - h(\tilde{p}) = \prod_{i=1}^{k} q_i$ is $1 - h(\hat{p}) = \prod_{i=1}^{k} (N_i - X_i)/N_i$, that is, the product of the observed proportion of failures of the individual components. However, if any component exhibits no failures in the experiment, then $h(\hat{p}) = 1$. Thus for data like $N_1 = 100, X_1 = 75$, $N_2 = 100$, $X_2 = 70$, $N_3 = 5$, $X_3 = 5$, we have $h(\hat{p}) = 1$, which is inherently a poor estimator.

6. Methods which may be used for arbitrary coherent systems. A. Madansky [24] treated the use of Wilk's likelihood ratio statistic for the construction of

confidence sets for the reliability function $h(\tilde{p})$ of series, parallel, and series-parallel systems. He described the use of Lagrange multipliers to accomplish the necessary constrained maximization. He also described what he called the linearization method, that is, expanding the maximum likelihood estimator $h(\hat{p})$ up to terms of first degree and thus obtaining the statistic

$$Z = (h(\hat{p}) - h(\tilde{p}))/S_n(\hat{\theta}) \quad , \tag{20}$$

where $S_n(\hat{\theta})$ is an estimator of the asymptotic standard derivation of $h(\hat{p})$. According to the usual large sample theory, Z is asymptotically normally distributed with zero mean and unit variance.

This line of investigation was continued by J. M. Myhre and S. C. Saunders [30], who compared the likelihood ratio with the linearization method and concluded that the likelihood ratio method was the more accurate of the two methods.

Myhre and Saunders [31] subsequently extended the work of Madansky on the likelihood ratio statistic to arbitrary coherent systems.

R. G. Easterling [11] proposed a modification of the linearization method in that the variance estimate $S_n^2(\hat{\theta})$ of (20) is replaced by

$$\hat{\sigma}^2 = h(\hat{p})\,(1 - h(\hat{p}))/\hat{n} \quad , \tag{21}$$

where $\hat{n}$ is called the pseudo-sample size. The estimate $\hat{n}$ is obtained by equating $\hat{\sigma}^2$ to the estimate of the asymptotic variance of the maximum likelihood estimate. The data is then to be regarded as a sample from a

binomial distribution with $\hat{x} = \hat{n}\, h(\hat{p})$ successes and $\hat{n}$ observations and this is used to obtain the lower confidence limit $\underline{L}_h$.

These methods all have the serious drawback that they perform very badly when the reliability is high, that is, near unity. Yet I feel that this is the region of the parameter space which is of greatest interest in applications.

A substantially different technique was proposed by Joan R. Rosenblatt [35]. Given the sample sizes $N_1, N_2, \ldots, N_k$ and the data $X_1 = x_1$, $X_2 = x_2$, $\ldots$, $X_k = x_k$, the author discussed simulation methods of statistical inference for $h(\tilde{p})$. One method of simulation is to select possible systems by sampling without replacement from each component. Then $\underline{N}$ systems are constructed and thus $\underline{N}$ values of $\varphi(\tilde{x})$ are obtained. Then the average $\sum_{j=1}^{\underline{N}} \varphi(\tilde{x}_j)/\underline{N}$ is an unbiased estimator of $h(\tilde{p})$ and by the central limit theorem, this statistic has an asymptotically normal distribution, permitting statistical inference to be easily carried out.

The author notes properly that this is an inefficient use of the available data. With the same data it is possible to construct $N = \prod_{i=1}^{k} N_i$ systems, obtaining thereby N values of $\varphi(\tilde{x})$. The average of these is an unbiased estimator of $h(\tilde{p})$. Further, she shows that this is a U-statistic in the sense of Hoeffding and that $\sqrt{N}\,(U - h(\tilde{p}))$ is asymptotically normally distributed whenever $N_i/N \to c_i$, $i = 1, 2, \ldots, k$. Also, it is shown that $\sigma_U^2 \leq \frac{1}{N}\left(h(\tilde{p})(1 - h(\tilde{p}))\right)$. Thus an asymptotic lower confidence limit for $h(\tilde{p})$ can easily be obtained. In general, this will be a conservative procedure.

7. Bayesian methods. There is an extensive literature on the use of Bayesian

methods in reliability analysis. In this section we provide a brief description of some of the literature in this area.

M. D. Springer and W. E. Thompson [36], [37] obtain the Bayesian solution for series and parallel systems by employing beta priors for each component. They employ the Mellin transform technique to combine the posterior distributions for the individual components and this is inverted to obtain the posterior distribution of $h(\tilde{p})$. The resulting formula is fairly intractable and the authors suggest the use of digital computers to evaluate this and thus to obtain confidence limits. An elementary discussion of Bayesian methods for series reliability using Beta priors for the individual components may be found in A. M. Breipohl, R. R. Prairie and W. J. Zimmer [9].

In W. E. Thompson and E. Y. Chang [38] the use of Chebyshev polynomials is suggested as a numerical method for evaluating the posterior distribution of the system reliability. The paper was written for the purpose of providing a Bayesian method for the reliability of parallel system when each component has an exponential lifetime distribution. However, the same technique can be employed in precisely the same manner whenever the posterior distribution of the reliability of the individual components has the form

$$(22) \qquad f(p_i) = \frac{(\beta_i+1)^{\alpha_i+1}}{\Gamma(\alpha_i+1)} p_i^{\beta_i} \ln(1/p_i)^{\alpha_i}, \quad \alpha_i > 1, \quad \beta_i > -1, \quad i = 1, 2, \ldots, k \ .$$

J. B. Parker [33] pointed out that one obtains different results if one chooses an uninformative prior for the system reliability than if one chooses the same uninformative prior for the reliability of each individual component. For example, a uniform prior might be an illustration of such a prior. Parker presents a table comparing component priors and system priors.

The use of Monte Carlo methods in conjunction with a Bayesian analysis is described in A. Hinojosa, D. V. Mastran and D. Spreen [17] and in D. V. Mastran [27]. A prior distribution is assumed for the reliability of each individual component. Then the experimental data is used to determine the posterior distribution. In [17] uniform priors are used and in [27] Beta priors are assumed. Then an estimate of $h(\tilde{p})$ is obtained by simulation, that is, a random reliability is chosen for each component by sampling from the respective posterior distributions. Repeating this, an estimate of the posterior distribution of the system reliability is obtained. A beta distribution is fitted to this empirical posterior by the method of moments and the lower confidence bound can then be determined.

In [17] the following problem is also discussed. Consider a series system with $k = 2$ and let $N_1 = N_2$ and assume that no failures have been observed. This can also be regarded as N trials of the complete system with no observed failures. The authors suggest that the second interpretation would give better reliability estimates, however, no theoretical justification for this remark is provided in the paper.

D. A. Berry [5] considered sampling plans which minimize the Bayes

risk, taking the sampling costs into account. Specifically he considers the loss function

$$(23)\quad L(h(\tilde{p}),\ h(\hat{p}),\ N_1, \ldots, N_k) = (h(\tilde{p}) - h(\hat{p}))^2 + \sum_{i=1}^{k} c_i N_i ,$$

where $c_i > 0$ is the cost of a single observation on the ith component. The specific details are carried out for series and parallel systems with $k = 2$ and uniform priors for the reliabilities of the individual components. This is extended to k components, $k > 2$ and priors of the form p^{m_i}.

In N. R. Mann and F. E. Grubbs [25], the posterior distribution of $-\ln h(\tilde{p})$ is approximated by a chi-square distribution to obtain lower confidence limits for series and parallel reliability. While computationally straightforward, the procedure is tedious to describe and the details are omitted. The same procedure is also described in the book by N. R. Mann, R. Schafer, and N. Singpurwalla [26]. The authors claim that their method provides excellent agreement with procedures known to possess some optimality property, such as Harris [15].

8. Decision theoretic models. To date, I have no evidence that decision theoretic techniques have been employed with the exception of Bayesian decision methods. Here, the only loss function that has been utilized is $L(h(\hat{p}),\ h(\tilde{p})) = (h(\hat{p}) - h(\tilde{p}))^2$ (see [5], [9]). I regard this as an inappropriate loss function. If the system reliability is .99, then on the average it should fail one time in 100, whereas if the system reliability is .999, it should fail about one time in 1000 and hence is ten times as good. Thus, the loss function should depend on how well one estimates $(1 - h(\tilde{p}))^{-1}$. Possibilities for

this are loss functions of the form

$$L(h(\hat{p}), h(\tilde{p})) = |(1 - h(\hat{p}))^{-1} - (1 - h(\tilde{p}))^{-1}|^{\alpha}, \quad \alpha > 0 .$$

It appears to me that this is a direction well worth considering.

9. Concluding remarks. Of necessity, this survey is far from complete. There are several reasons for this. Much of the literature of reliability theory is contained in proprietary publications and is not in the usual open literature. Hence, many articles could not be obtained easily. Similar inaccessibility applies to papers appearing in a number of foreign journals which could not be easily obtained. In addition, some readily accessible articles have surely been inadvertently overlooked.

Finally, it is clear that there are an enormous number of methods available and, to the best of my knowledge, there has been no definitive work to date which provides a reliable and comprehensive comparison between the different techniques. Many papers compare numerical values for the lower confidence limit for $h(\tilde{p})$ for several techniques, but this is an incomplete comparison, since the true confidence coefficient has only been approximated by the stated confidence coefficient, and the former is in general unknown.

REFERENCES

[1] Anonymous, A method of finding confidence limits for P_T.

[2] R. E. Barlow and F. Proschan, Mathematical Theory of Reliability, John Wiley and Sons, New York, 1965.

[3] R. E. Barlow and F. Proschan, Importance of system components and fault tree events, Stochastic Processes and Their Applications, 3 (1975), 153-173.

[4] R. E. Barlow and F. Proschan, Statistical Theory of Reliability and Life Testing, Probability Models, Holt, Rinehart and Winston, Inc., New York, 1975.

[5] D. A. Berry, Optimal sampling schemes for estimating reliability by testing components – I: fixed sample sizes, J. Amer. Statist. Assoc., 69(1974), 485-491.

[6] Z. W. Birnbaum, On the importance of different components in a multi-component system, in Multivariate Analysis, II, P. R. Krishnaiah, Ed., Academic Press, New York, 1969, 581-592.

[7] Z. W. Birnbaum, J. D. Esary and S. C. Saunders, Multi-component systems and their reliability, Technometrics, 3 (1961), 55-77.

[8] L. N. Bol'shev and E. A. Loginov, Internal estimates in the presence of noise, Theory of Probability and its Applications, 11 (1966), 82-94.

[9] A. M. Breipohl, R. R. Prairie and W. J. Zimmer, A consideration of the Bayesian approach in reliability evaluation, I. E. E. E. Trans. on Reliability, R-14 (1965), 107-113.

[10] R. J. Buehler, Confidence limits for the product of two binomial parameters, J. Amer. Statist. Assoc., 52(1957), 482-493.

[11] R. G. Easterling, Approximate confidence limits for series reliability, J. Amer. Statist. Assoc., 67(1972), 220-222.

[12] J. L. Epstein, Upper confidence limits on the product of two binomial parameters, Cornell University Master of Science Thesis, 1967.

[13] N. R. Garner and R. W. Vail, Jr., Confidence limits for systems reliability, Military Systems Design, 7 (1961).

[14] B. V. Gnedenko, Yu. K. Belyayev and A. D. Solov'yev, Mathematical Methods of Reliability Theory, Nauka 1965, (English translation, Academic Press, New York, 1969).

[15] B. Harris, Hypothesis testing and confidence intervals for products and quotients of Poisson parameters with applications to reliability, J. Amer. Statist. Assoc., 66 (1971), 609-613.

[16] B. Harris and A. P. Soms, Tables of the two factor and three factor generalized incomplete modified Bessel distributions, in Selected Tables in Mathematical Statistics, American Mathematical Society, Providence, 1975.

[17] A. Hinojosa, D. V. Mastran and D. Spreen, Monte Carlo approach to reliability statements based on attribute data, in A Tri-Service Approach for Reliability Assessment, Naval Ammunition Depot, Oahu, Hawaii, 1969.

[18] J. P. Lipp, Topology of switching elements vs. reliability, IRE Trans. on Reliability and Quality Control, 6 (1957), 21-33.

[19] D. K. Lloyd and M. Lipow, Reliability: Management, Methods, and Mathematics, Prentice Hall, Englewood Cliffs, New Jersey, 1962.

[20] C. Mack, Confidence limits for the product of binomial parameters and related problems, New J. Statistics and Operations Research, 1 (1965), 1-16.

[21] C. Mack, Confidence limits for the product of two binomial parameters when observed defectives are few: I, New J. Statistics and Operations Research, 3 (1967), 3-13.

[22] C. Mack, Confidence limits for the product of binomial parameters when observed failures are few: II, New J. Statistics and Operations Research, 3 (1967), 15-24.

[23] C. Mack, Tables of confidence limits for the product $p_1 p_2$ of two binomial parameters when observed successes (= defectives) are few or moderate and samples are large or medium, New J. Statistics and Operations Research, 3 (1967), 26-35.

[24] A. Madansky, Approximate confidence limits for the reliability of series and parallel systems, Technometrics, 7 (1965), 495-503.

[25] N. R. Mann and F. E. Grubbs, Approximately optimum confidence bounds for system reliability based on component test data, Technometrics, 16 (1974), 335-347.

[26] N. R. Mann, R. E. Schafer, and N. D. Singpurwalla, Methods for Statistical Analysis of Reliability and Life Data, John Wiley and Sons, New York, 1974.

[27] D. V. Mastran, A Bayesian approach for assessing the reliability of Air Force re-entry systems, Proc. of the ASME Reliability and Maintainability Symposium, Gordon and Breach, New York, 1969.

[28] R. A. Mirniy and A. D. Solov'yev, Estimation of the reliability of a system from the results of tests of its components, in Kibernetiky na Sluzhby Kommunizmy, 2, Energiya, Moscow, 1964.

[29] E. F. Moore and C. E. Shannon, Reliable circuits using less reliable relays, J. of the Franklin Institute, 262(1956), 191-208.

[30] J. M. Myhre and S. C. Saunders, Comparison of two methods of obtaining approximate confidence intervals for system reliability, Technometrics, 10(1968), 37-49.

[31] J. M. Myhre and S. C. Saunders, On confidence limits for the reliability of systems, Ann. Math. Statist., 39(1968), 1463-1472.

[32] F. Nishime, Techniques for ... the establishment of confidence limits for the estimated reliability, Space Technology Laboratories (1959).

[33] J. B. Parker, Bayesian prior distributions for multi-component systems, Naval Research Logistics Quarterly, 19 (1972), 509-515.

[34] I. V. Pavlov, A confidence estimate of system reliability from component testing results, Izvestiya Akad. Nauk. Tech. Kibernetiky, 3(1973), 52-61.

[35] J. R. Rosenblatt, Confidence limits for the reliability of complex systems, in Statistical Theory of Reliability, M. Zelen, Ed., University of Wisconsin Press, Madison, 1963.

[36] M. D. Springer and W. E. Thompson, Bayesian confidence limits for the product of N binomial parameters, Biometrika, 53 (1966), 611-613.

[37] M. D. Springer and W. E. Thompson, Bayesian confidence limits for system reliability, Proc. A. S. M. E. Reliability and Maintainability Symposium, Gordon and Breach, New York, 1969.

[38] W. E. Thompson and E. Y. Chang, Bayesian confidence limits for reliability of redundant systems, Technometrics, 17(1975), 89-93.

[39] K. A. Weaver, The determination of the lower confidence limit on the reliability of a serial system, Proc. of the NATO Conference on Reliability Theory, Turin, 1969, Gordon and Breach, New York, 1971.

[40] A. Winterbottom, Lower limits for series system reliability in the attribute case, Research Memorandum 8, The City University, London.

[41] A. Winterbottom, Lower limits for series system reliability from binomial data, J. Amer. Statist. Assoc., 69 (1974), 782-788.

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